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A characterization of the weighted version of McEliece–Rodemich–Rumsey–Schrijver number based on convex quadratic programming

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For any graph G, Luz and Schrijver [A convex quadratic characterization of the Lovász theta number, *SIAM J. Discrete Math.* **19**(2) (2005) 382–387] introduced a characterization of the Lovász number $\vartheta(G)$ based on convex quadratic programming. A similar characterization is now established for the weighted version of the number $\vartheta'(G)$, independently introduced by McEliece, Rodemich, and Rumsey [The Lovász bound and some generalizations, *J. Combin. Inform. Syst. Sci.* **3** (1978) 134–152] and Schrijver [A Comparison of the Delsarte and Lovász bounds, *IEEE Trans. Inform. Theory* **25**(4) (1979) 425–429]. Also, a class of graphs for which the weighted version of $\vartheta'(G)$ coincides with the weighted stability number is characterized.

Keywords: Lovász number; McEliece–Rodemich–Rumsey–Schrijver number; maximum weight stable set; combinatorial optimization; graph theory; quadratic programming.

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1. Introduction

Let $G = (V, E)$ be a simple undirected graph where $V = \{1, 2, \ldots, n\}$ and E are respectively the vertex and edge sets. Throughout this paper it will be supposed that G has at least one edge (i.e., E is nonempty) and the notation $ij \in E$ will be used to denote the edge linking vertices i and j of V. The adjacency matrix of G is the symmetric matrix $A_G \in \mathbb{R}^{n\times n}$ whose entries (i, j) are equal to 1 if $ij \in E$ and 0 otherwise. By replacing some or all of the ones of A_G with any real numbers such that the resulting matrix remains nonnull and symmetric, we obtain a so-called *weighted adjacency matrix* of G. Furthermore, an *extended weighted adjacency matrix* of G can be obtained if some or all entries corresponding to edges $ij \notin E$ in a weighted adjacency matrix of G are replaced with negative real numbers such that the resulting matrix remains nonnull and symmetric.

A graph $G = (V, E)$ is said to be a weighted graph if each vertex $i \in V$ has assigned a positive weight $w_i \in \mathbb{R}^+$. Denoting by $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ the vector of vertex weights, the weighted graph will be represented by (G, w). In the sequel there will be no distinction between the (extended or not) weighted adjacency matrices of G and (G, w).

A stable set (or independent set) of $G = (V, E)$ is a subset of vertices of V whose elements are pairwise nonadjacent. The stability number (or independence number) of G is defined as the cardinality of a largest stable set and is usually denoted by $\alpha(G)$. A maximum stable set of G is a stable set with $\alpha(G)$ vertices. More generally, if (G, w) is a weighted graph, we can talk about a maximum weight stable set of G which is defined as a stable set for which the sum of vertex weights is maximum. This maximum sum is referred to as the weighted stability number of (G, w) and will be denoted by $\alpha(G, w)$.

The problem of finding $\alpha(G)$ is NP-*hard* and the same happens with $\alpha(G, w)$, since this number equals $\alpha(G)$ in the unweighted case, i.e., when all vertex weights are equal to one. However several ways of approaching those numbers have been proposed in the literature (see, for example, [1, 5, 8, 11, 17] and the surveys [2, 19]).

For any graph G with at least one edge, the upper bound $\upsilon(G)$ on $\alpha(G)$ defined as the optimal value of the following convex quadratic programming problem was introduced in [12]:

$$P(G) \quad \upsilon(G) = \max\{2e^T x - x^T (H + I)x : x \geq 0\},$$

where, hereinafter, e denote the $n \times 1$ all ones vector, T stands for the transposition operation, I is the identity matrix of order n, $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, $x \geq 0$ means that all coordinates x_i of vector x are non-negative and $H = A_G/(-\lambda_{\min}(A_G))$. (In what follows, $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ will denote respectively the smallest and the largest eigenvalues of a matrix M; also, all considered vectors are column vectors.)

Since G has at least one edge, A_G is indefinite as its trace is zero. Hence $\lambda_{\min}(H) = -1$ and this guarantees the convexity of $P(G)$ because $H + I$ is positive semidefinite. Consequently, $\upsilon(G)$ can be computed in polynomial time.

The graphs that satisfy $\alpha(G) = \upsilon(G)$ were introduced in [12] and subsequently studied in [3, 4, 13]. They are currently known as graphs with convex-QP stability number (or convex-QP graphs, where QP means quadratic programming).

The upper bound $\upsilon(G)$ was extended to the weighted case in [14]. The obtained extension, denoted here by $\upsilon(G, w)$, constitutes an upper bound on $\alpha(G, w)$ which, similarly to the unweighted case, can be computed by solving a quadratic programming problem:

$$\begin{aligned}\alpha(G, w) &\leq \upsilon(G, w)\\ &= \max\left\{2w^T x - x^T \left(\frac{A_G}{-\lambda_{\min}(W^{-1/2} A_G W^{-1/2})} + W\right) x : x \geq 0\right\}, \qquad (1)\end{aligned}$$

where W is the diagonal matrix whose main diagonal elements are the coordinates of w (i.e., $W = \operatorname{diag}(w_1, \ldots, w_n)$) and $W^{-1/2}$ denote the inverse matrix of $W^{1/2} =$

$\mathrm{diag}(\sqrt{w_1}, \ldots, \sqrt{w_n})$. Note that the problem in (1) is equivalent to the following:

$$\upsilon(G, w) = \max\left\{2\sqrt{w}^T x - x^T\left(\frac{W^{-1/2}A_G W^{-1/2}}{-\lambda_{\min}(W^{-1/2}A_G W^{-1/2})} + I\right)x : x \geq 0\right\}, \qquad (2)$$

where $\sqrt{w} = (\sqrt{w_1}, \ldots, \sqrt{w_n})^T$ is the vector whose coordinates are the square roots of vector w coordinates. In fact, substituting $y = W^{1/2}x$ for x in (1) we obtain the problem in (2) if after the substitution we continue using x instead of y for denoting the variables vector. Note also that $W^{-1/2}A_G W^{-1/2}$ is indefinite (since the same happens with A_G) and that $\frac{W^{-1/2}A_G W^{-1/2}}{-\lambda_{\min}(W^{-1/2}A_G W^{-1/2})} + I$ is positive semidefinite. Then the quadratic programming problem in (2) is convex and hence can be solved in polynomial time.

The Lovász number, usually denoted by $\vartheta(G)$, was introduced in [10] and is probably the most famous upper bound on $\alpha(G)$. It can be computed in polynomial time as proved by Grötschel, Lovász and Schrijver [6] and many characterizations of $\vartheta(G)$ are known, some of them are given in [10] (see [8, 9] for a detailed treatment of the subject).

Luz and Schrijver [15] introduced another characterization of $\vartheta(G)$ which is based on convex quadratic programming. As a matter of fact, with the aim of relating the upper bounds $\upsilon(G)$ and $\vartheta(G)$, they considered the family of convex quadratic problems,

$$P(G, C) \quad \upsilon(G, C) = \max\{2e^T x - x^T(H_C + I)x : x \geq 0\},$$

where C is a weighted adjacency matrix of G and $H_C = C/-\lambda_{\min}(C)$. As the adjacency matrix A_G, matrix C is also indefinite and consequently, since $\lambda_{\min}(H_C) = -1$, all problems $P(G, C)$ are convex. Observe in addition that $\upsilon(G, A_G) = \upsilon(G)$ and hence $P(G)$ belongs to the family of $P(G, C)$ problems.

In consequence, the following characterization of $\vartheta(G)$ based on convex quadratic programming was given (see [15, Theorem 4.2]):

$$\vartheta(G) = \min_C \upsilon(G, C) = \min_C \max_{x \geq 0}\{2e^T x - x^T(H_C + I)x\}, \qquad (3)$$

where C is a weighted adjacency matrix of G.

The number usually denoted by $\vartheta'(G)$ was independently introduced by McEliece, Rodemich, and Rumsey [16] and Schrijver [18]. It is also an upper bound on the stability number $\alpha(G)$ which is generally sharper than $\vartheta(G)$ since the following inequalities hold for each graph G (see [18]):

$$\alpha(G) \leq \vartheta'(G) \leq \vartheta(G). \qquad (4)$$

Two characterizations of $\vartheta'(G)$ are presented below which can be also seen in [18]. The first one is:

$$\vartheta'(G) = \min_{M \in \mathcal{M}(G)} \lambda_{\max}(M), \qquad (5)$$

where the minimum is taken over the set $\mathcal{M}(G)$ of all symmetric matrices $M = [m_{ij}] \in \mathbb{R}^{n \times n}$ such that $m_{ij} = 1$ if $i = j$ and $m_{ij} \geq 1$ if $ij \notin E$.

The second characterization of $\vartheta'(G)$ is dual of the previous one:

$$\vartheta'(G) = \max_{B\in\mathcal{B}(G)} e^T Be, \tag{6}$$

where the maximum is taken over the set $\mathcal{B}(G)$ of all non-negative symmetric positive semidefinite matrices $B = [b_{ij}] \in \mathbb{R}^{n\times n}$ such that $b_{ij} = 0$ for $ij \in E$ and $\mathrm{Tr}(B) = 1$. ($\mathrm{Tr}(B)$ denotes the trace of B.)

In this paper the characterization (3) is extended to the weighted version of $\vartheta'(G)$, which, for any weighted graph (G, w), will be denoted by $\vartheta'(G, w)$. We begin, in Sec. 2, with some $\vartheta'(G, w)$ definitions and then, in Sec. 3, the new characterization of $\vartheta'(G, w)$ based on convex quadratic programming is deduced. In Sec. 4, the class of weighted graphs (G, w) for which $\alpha(G, w) = \vartheta'(G, w)$ is characterized and an example of such a graph is presented.

2. Defining $\vartheta'(G, w)$

Recall the notations $\sqrt{w}$, W, $W^{1/2}$ and $W^{-1/2}$ set out in Sec. 1 for weighted graphs (G, w). The weighted version of Lovász number, usually denoted by $\vartheta(G, w)$, was introduced by Grötschel, Lovász and Schrijver [6] and studied in detail in [7–9]. In a similar way, the weighted version of $\vartheta'(G)$, i.e., $\vartheta'(G, w)$, is defined here by extending the characterization (5) as follows:

$$\vartheta'(G, w) = \min_{M\in\mathcal{M}(G)} \lambda_{\max}(W^{1/2}MW^{1/2}), \tag{7}$$

where, as before, $\mathcal{M}(G)$ is the set of all symmetric matrices $M = [m_{ij}] \in \mathbb{R}^{n\times n}$ such that $m_{ij} = 1$ if $i = j$ and $m_{ij} \geq 1$ if $ij \notin E$.

Since we are assuming that G has at least one edge, we eliminate the matrix ee^T from $\mathcal{M}(G)$. In fact, if $\vartheta'(G, w) = \lambda_{\max}(W^{1/2}ee^TW^{1/2}) = \lambda_{\max}(\sqrt{w}\sqrt{w}^T) = e^Tw$, we would obtain the largest possible value of $\vartheta'(G, w)$ which is only attained if (G, w) has no edge (because, as it will be seen below, $\vartheta'(G, w) \leq \vartheta(G, w)$ and the largest possible value of $\vartheta(G, w)$ is e^Tw, see [9]).

Let M be one of the above symmetric matrices. As $M \neq ee^T$ we have that $Q = ee^T - M \neq 0$ is an extended weighted adjacency matrix of (G, w). Consequently, setting $M = ee^T - Q$, the characterization (7) can be written in the form

$$\vartheta'(G, w) = \min_{Q} \lambda_{\max}(\sqrt{w}\sqrt{w}^T - W^{1/2}QW^{1/2}), \tag{8}$$

where Q is an extended weighted adjacency matrix of (G, w).

We can also have a characterization of $\vartheta'(G, w)$ which is dual of (8) and generalizes (6):

$$\vartheta'(G, w) = \max_{B\in\mathcal{B}(G)} \sqrt{w}^T B\sqrt{w}, \tag{9}$$

where, as before, $\mathcal{B}(G)$ is the set of all non-negative symmetric positive semidefinite matrices $B = [b_{ij}] \in \mathbb{R}^{n\times n}$ such that $b_{ij} = 0$ for $ij \in E$ and $\mathrm{Tr}(B) = 1$.

Using (9), the inequalities (4) can be easily generalized for weighted graphs.

Proposition 1. *Let (G, w) be a weighted graph. Then $\alpha(G, w) \le \vartheta'(G, w) \le \vartheta(G, w)$.*

Proof. Note first that if the nonnegativity of $\mathcal{B}(G)$ matrices is relaxed, the maximum in (9) becomes $\vartheta(G, w)$ (see for example [8]). Hence, $\vartheta'(G, w) \le \vartheta(G, w)$. On the other hand, considering the matrix $B = \frac{1}{\alpha(G,w)} xx^T$, where x is defined by $x_i = \sqrt{w_i}$ if $i \in S$ and $x_i = 0$ otherwise with S being a maximum weighted stable set of (G, w), $B \in \mathcal{B}(G)$ and $\sqrt{w}^T B \sqrt{w} = \alpha(G, w)$. Consequently, $\alpha(G, w) \le \vartheta'(G, w)$. □

3. The New Characterization of $\vartheta'(G, w)$

Our first aim is to relate $\vartheta'(G, w)$ to a family of quadratic upper bounds on $\alpha(G, w)$ which includes the upper bound $\upsilon(G, w)$ given in (2). Thus, associated to a weighted graph (G, w), consider the matrices

$$H_{C,w} = \frac{W^{-1/2} C W^{-1/2}}{-\lambda_{\min}(W^{-1/2} C W^{-1/2})},$$

where $C = [c_{ij}] \in \mathbb{R}^{n \times n}$ is any extended weighted adjacency matrix of G, and the convex quadratic programming problems

$$P(G, w, C) \quad \upsilon(G, w, C) = \max\{2\sqrt{w}^T x - x^T(H_{C,w} + I)x : x \ge 0\}.$$

Note that $\upsilon(G, w, C)$ generalizes the upper bound $\upsilon(G, w)$ given in (2) since $\upsilon(G, w) = \upsilon(G, w, A_G)$.

We show first that $\upsilon(G, w, C)$ is an upper bound on the weighted stability number $\alpha(G, w)$.

Proposition 2. *Let (G, w) be a weighted graph with at least one edge. For any extended weighted adjacency matrix $C = [c_{ij}]$ of (G, w), $\upsilon(G, w, C)$ is the optimal value of a convex quadratic programming problem and verifies $\alpha(G, w) \le \upsilon(G, w, C)$, i.e., $\upsilon(G, w, C)$ is an upper bound on $\alpha(G, w)$.*

Proof. The matrix $H_{C,w}$ is indefinite since its trace is null and not all its entries are null. Thus $\lambda_{\min}(H_{C,w}) = -1$ and this guarantees the convexity of $P(G, w, C)$ because $H_{C,w} + I$ is positive semidefinite.

To see that $\upsilon(G, w, C)$ is an upper bound on $\alpha(G, w)$ for all extended weighted adjacency matrices C, let S be a maximum weight stable set of (G, w) and x be the vector defined by $x_i = \sqrt{w_i}$ if $i \in S$ and $x_i = 0$ otherwise. Since x is a feasible solution of $P(G, w, C)$, we have

$$\begin{aligned} \upsilon(G, w, C) &\ge 2\sqrt{w}^T x - x^T x - x^T H_{C,w}\, x \\ &= 2\alpha(G, w) - \alpha(G, w) - \frac{1}{-\lambda_{\min}(W^{-1/2} C W^{-1/2})} \sum_{i,j} \frac{c_{ij}}{\sqrt{w_i w_j}} x_i x_j \end{aligned}$$

$$= \alpha(G,w) - \frac{1}{-\lambda_{\min}(W^{-1/2}CW^{-1/2})}$$
$$\times \left(\sum_{i\in V} \frac{c_{ii}}{w_i} x_i^2 + 2 \sum_{ij\in E} \frac{c_{ij}}{\sqrt{w_i w_j}} x_i x_j + 2 \sum_{ij\notin E} \frac{c_{ij}}{\sqrt{w_i w_j}} x_i x_j \right).$$

As $\lambda_{\min}(W^{-1/2}CW^{-1/2}) < 0$, $c_{ii} = 0$ for all $i \in V$, $x_i x_j = 0$ if $ij \in E$ and $c_{ij} \leq 0$ if $ij \notin E$, the inequality $\upsilon(G,w,C) \geq \alpha(G,w)$ is true for all extended weighted adjacency matrices C of (G,w). □

We now relate $\vartheta'(G,w)$ with the convex quadratic upper bounds $\upsilon(G,w,C)$. First, it is proved that $\vartheta'(G,w)$ is not worse than any $\upsilon(G,w,C)$ bound.

Theorem 1. *Let (G,w) be a weighted graph with at least one edge. Then, for any extended weighted adjacency matrix C of (G,w), we have $\vartheta'(G,w) \leq \upsilon(G,w,C)$.*

Proof. Let C be an extended weighted adjacency matrix of (G,w) and suppose that $P(G,w,C)$ is not unbounded otherwise the theorem is true.

Let x be an optimal solution of $P(G,w,C)$. The Karush–Kuhn–Tucker conditions applied to this problem guarantee that the following conditions are true:

$$x \geq 0, \quad (H_{C,w} + I)x \geq \sqrt{w}, \quad \text{and} \quad x^T(H_{C,w} + I)x = \sqrt{w}^T x = \upsilon(G,w,C). \tag{10}$$

As $H_{C,w} + I$ is positive semidefinite we can write $H_{C,w} + I = U^T U$. Denoting the columns of U by $u_1, \ldots, u_n$, define a matrix $M = [m_{ij}] \in \mathbb{R}^{n\times n}$ such that

$$m_{ij} = 1 - \frac{u_i^T u_j}{(c^T u_i)(c^T u_j)} \quad \text{if } i \neq j,$$
$$m_{ii} = 1,$$

where $c = \upsilon_w^{-1/2} Ux$ (we use υ_w to abbreviate $\upsilon(G,w,C)$). By (10), we have

$$U^T c = \upsilon_w^{-1/2} U^T U x \geq \upsilon_w^{-1/2} \sqrt{w}; \tag{11}$$

hence $M \in \mathcal{M}(G)$ since it is symmetric, $m_{ii} = 1$ and $m_{ij} \geq 1$ if $ij \notin E$ (as $u_i^T u_j \leq 0$ if $ij \notin E$ and $c^T u_i > 0$ for all i by (11)).

On the other hand, (11) implies that $\frac{w_i}{(c^T u_i)^2} \leq \upsilon_w$, for all i, and from (10) we can conclude that $c^T c = \upsilon_w^{-1} x^T (H_{C,w} + I)x = 1$. Thus we can write

$$-\sqrt{w_i} m_{ij} \sqrt{w_j} = \sqrt{w_i} \left(c - \frac{u_i}{c^T u_i} \right)^T \left(c - \frac{u_j}{c^T u_j} \right) \sqrt{w_j}$$

and

$$\upsilon_w - \sqrt{w_i} m_{ii} \sqrt{w_i} = w_i \left(c - \frac{u_i}{c^T u_i} \right)^2 + \upsilon_w - \frac{w_i}{(c^T u_i)^2}.$$

These equations guarantee that matrix $\upsilon_w I - W^{1/2} M W^{1/2}$ is positive semidefinite and hence $\lambda_{\max}(W^{1/2} M W^{1/2}) \leq \upsilon_w$. Finally, by (7), we conclude $\vartheta'(G,w) \leq \upsilon(G,w,C)$ as desired. □

The next theorem establishes the characterization of $\vartheta'(G,w)$ based on convex quadratic programming. The proof given below generalizes proof of [15, Theorem 4.2]. Along it and in the rest of this paper, to simplify the notation, we will sometimes use ϑ'_w instead of $\vartheta'(G,w)$.

Theorem 2. *Let (G,w) be a weighted graph with at least one edge. If Q attains the optimum in* (8) *then $\vartheta'(G,w) = \upsilon(G,w,C)$, where $C = WQW$.*

Consequently, the following characterization of $\vartheta'(G,w)$ is valid:

$$\vartheta'(G,w) = \min_{C} \upsilon(G,w,C) = \min_{C} \max_{x \geq 0} \{2\sqrt{w}^T x - x^T(H_{C,w} + I)x\}, \tag{12}$$

where $C = WQW$ is an extended weighted adjacency matrix of (G,w).

Proof. Let Q be an extended weighted adjacency matrix of (G,w) attaining the optimum in (8). As $\vartheta'(G,w) = \lambda_{\max}(\sqrt{w}\sqrt{w}^T - W^{1/2}QW^{1/2}) \geq -\lambda_{\min}(W^{1/2} \times QW^{1/2})$, we will divide into two cases the proof of the equality $\vartheta'(G,w) = \upsilon(G,w,C)$, where $C = WQW$.

Case 1: $\vartheta'(G,w) = -\lambda_{\min}(W^{1/2}QW^{1/2})$.

Let x attain the optimum in $P(G,w,C)$. Then, using the positive semidefiniteness of $I - (\vartheta'_w)^{-1}(\sqrt{w}\sqrt{w}^T - W^{1/2}QW^{1/2})$, we have

$$\begin{aligned}
&\upsilon(G,w,C)\\
&\quad = 2\sqrt{w}^T x - x^T(H_{C,w} + I)x = 2\sqrt{w}^T x - x^T\left(\frac{W^{1/2}QW^{1/2}}{-\lambda_{\min}(W^{1/2}QW^{1/2})} + I\right)x\\
&\quad = 2\sqrt{w}^T x - x^T[I + (\vartheta'_w)^{-1}W^{1/2}QW^{1/2} - (\vartheta'_w)^{-1}\sqrt{w}\sqrt{w}^T]x - (\vartheta'_w)^{-1}(\sqrt{w}^T x)^2\\
&\quad = 2\sqrt{w}^T x - x^T[I - (\vartheta'_w)^{-1}(\sqrt{w}\sqrt{w}^T - W^{1/2}QW^{1/2})]x - (\vartheta'_w)^{-1}(\sqrt{w}^T x)^2\\
&\quad \leq 2\sqrt{w}^T x - (\vartheta'_w)^{-1}(\sqrt{w}^T x)^2 \leq \vartheta'_w,
\end{aligned}$$

since $[(\vartheta'_w)^{1/2} - (\vartheta'_w)^{-1/2}\sqrt{w}^T x]^2 \geq 0$. So, by Theorem 1, we have $\vartheta'(G,w) = \upsilon(G,w,C)$ for this case.

Case 2: $\vartheta'(G,w) > -\lambda_{\min}(W^{1/2}QW^{1/2})$.

Let B attain the optimum in (9). Since $\vartheta'_w I - \sqrt{w}\sqrt{w}^T + W^{1/2}QW^{1/2}$ is positive semidefinite we have, by the Fejer's trace theorem,

$$\mathrm{Tr}[B(\vartheta'_w I - \sqrt{w}\sqrt{w}^T + W^{1/2}QW^{1/2})] \geq 0.$$

On the other hand,

$$\begin{aligned}
&\mathrm{Tr}[B(\vartheta'_w I - \sqrt{w}\sqrt{w}^T + W^{1/2}QW^{1/2})]\\
&\qquad = \vartheta'_w \mathrm{Tr}(B) - \mathrm{Tr}(B\sqrt{w}\sqrt{w}^T) + \mathrm{Tr}[B(W^{1/2}QW^{1/2})]\\
&\qquad = \vartheta'_w - \vartheta'_w + \mathrm{Tr}[B(W^{1/2}QW^{1/2})]\\
&\qquad = \mathrm{Tr}[B(W^{1/2}QW^{1/2})] \leq 0,
\end{aligned}$$

as B is non-negative and Q is an extended weighted adjacency matrix of (G, w). Thus

$$\mathrm{Tr}[B(\vartheta'_w I - \sqrt{w}\sqrt{w}^T + W^{1/2}QW^{1/2})] = 0$$

and, using the positive semidefiniteness of $\vartheta'_w I - \sqrt{w}\sqrt{w}^T + W^{1/2}QW^{1/2}$ and B, we have $B(\vartheta'_w I - \sqrt{w}\sqrt{w}^T + W^{1/2}QW^{1/2}) = O$, where O denotes the null matrix. Thus, the column space of B is orthogonal to the column space of $\vartheta'_w I - \sqrt{w}\sqrt{w}^T + W^{1/2}QW^{1/2}$.

The inequality $\vartheta'_w(G) > -\lambda_{\min}(W^{1/2}QW^{1/2})$ implies that $\lambda_{\min}(\vartheta'_w I + W^{1/2}QW^{1/2}) > 0$ and hence $\mathrm{rank}(\vartheta'_w I + W^{1/2}QW^{1/2}) = n$. Then $\mathrm{rank}(\vartheta'_w I - \sqrt{w}\sqrt{w}^T + W^{1/2}QW^{1/2}) \geq n-1$ and by the column spaces orthogonality, $\mathrm{rank}(B) \leq 1$. As $\mathrm{Tr}(B) = 1$, $\mathrm{rank}(B) = 1$ and then $B = (\vartheta'_w)^{-1}xx^T$ for some vector x whose support is a stable set S. Since $\sqrt{w}^T B\sqrt{w} = \vartheta'_w$ and $\mathrm{Tr}(B) = 1$, we can choose $x \geq 0$ and thus we have $\sqrt{w}^T x = x^T x = \vartheta'_w$. In addition, x is given by

$$x_i = \begin{cases} \sqrt{w_i} & \text{if } i \in S, \\ 0 & \text{if } i \notin S, \end{cases} \tag{13}$$

i.e., $x = W^{1/2}y$, where y is a characteristic vector of S. (To see this, let $z = (z_i)_{i=1,\ldots,n}$ be a vector such that $z_i = \sqrt{w_i}$ if $i \in S$ and $z_i = 0$ if $i \notin S$. Then, $z^T x = \sqrt{w}^T x = \vartheta'_w$ and, using the Cauchy–Schwarz inequality, $(z^T x)^2 \leq (x^T x)(z^T z)$. So $\vartheta'_w \leq z^T z = \sum_{i \in S} w_i$, and by the maximality of ϑ'_w we have $z^T z = \vartheta'_w$. Hence, the Cauchy–Schwarz inequality is satisfied with equality and this implies $x = z$.)

Using once more the orthogonality of the column spaces of B and $\vartheta'_w I - \sqrt{w}\sqrt{w}^T + W^{1/2}QW^{1/2}$, we conclude that $(\sqrt{w}\sqrt{w}^T - W^{1/2}QW^{1/2})x = \vartheta'_w x$, and hence $W^{1/2}QW^{1/2}x = \vartheta'_w(\sqrt{w} - x)$. Then x satisfies the Karush–Kuhn–Tucker conditions associated with $P(G, w, C)$ (recall (10)) as:

- $x \geq 0$;
- $(H_{C,w} + I)x = (\frac{W^{1/2}QW^{1/2}}{-\lambda_{\min}(W^{1/2}QW^{1/2})} + I)x = \frac{W^{1/2}QW^{1/2}x}{-\lambda_{\min}(W^{1/2}QW^{1/2})} + x = \frac{\vartheta'_w}{-\lambda_{\min}(W^{1/2}QW^{1/2})}(\sqrt{w} - x) + x \geq \sqrt{w}$, since $\vartheta'_w \geq -\lambda_{\min}(W^{1/2}QW^{1/2})$; and
- $x^T(H_{C,w} + I)x = \frac{\vartheta'_w}{-\lambda_{\min}(W^{1/2}QW^{1/2})}x^T(\sqrt{w} - x) + x^T x = \vartheta'_w$, since $x^T(\sqrt{w} - x) = 0$.

Consequently, by the positive semidefiniteness of $H_{C,w} + I$, the equality $\vartheta'(G, w) = \upsilon(G, w, C)$ is also true for Case 2.

Finally, the proved equality, the definition of Q and Theorem 1 imply the characterization (12). □

4. A Class of Graphs for Which $\alpha(G, w) = \vartheta'(G, w)$

As a consequence of Theorem 2, a necessary and sufficient condition that characterizes the weighted graphs (G, w) for which $\alpha(G, w) = \vartheta'(G, w)$ is given.

Theorem 3. *Let (G,w) be a weighted graph with at least one edge and Q an extended weighted adjacency matrix such that $C = WQW$ attains the optimum in* (12). *Then $\alpha(G,w) = \vartheta'(G,w)$ if and only if there is a maximum weight stable set S of (G,w) for which the following conditions hold*:

$$\sum_{j\in S} c_{ij} = 0, \quad \forall\, i \in S \tag{14}$$

and

$$-\lambda_{\min}(W^{1/2}QW^{1/2}) \le \frac{1}{w_i}\sum_{j\in S} c_{ij}, \quad \forall\, i \notin S, \tag{15}$$

where c_{ij} denotes the entry (i,j) of matrix C.

Proof. Theorem 2 allows to conclude that $\alpha(G,w) = \vartheta'(G,w)$ if and only if $\alpha(G,w) = \upsilon(G,w,C)$, where $C = WQW$. We will prove that $\alpha(G,w) = \upsilon(G,w,C)$ holds if and only if conditions (14) and (15) are satisfied.

To begin with, suppose that these conditions hold for a maximum weight stable set S of (G,w). If $x = W^{1/2}y$, where y is the characteristic vector of S, we have

$$\begin{aligned}
&2\left(\frac{W^{-1/2}CW^{-1/2}}{-\lambda_{\min}(W^{-1/2}CW^{-1/2})} + I\right)x \\
&\quad = 2\frac{W^{-1/2}Cy}{-\lambda_{\min}(W^{-1/2}CW^{-1/2})} + 2W^{1/2}y \\
&\quad = \begin{cases} 2\dfrac{1}{\sqrt{w_i}}\dfrac{\sum_{j\in S} c_{ij}}{-\lambda_{\min}(W^{-1/2}CW^{-1/2})} + 2\sqrt{w_i} & \text{if } i \in S, \\[2ex] 2\dfrac{1}{\sqrt{w_i}}\dfrac{\sum_{j\in S} c_{ij}}{-\lambda_{\min}(W^{-1/2}CW^{-1/2})} & \text{if } i \notin S. \end{cases}
\end{aligned}$$

Taking into account condition (14), we can write

$$2\left(\frac{W^{-1/2}CW^{-1/2}}{-\lambda_{\min}(W^{-1/2}CW^{-1/2})} + I\right)x = \begin{cases} 2\sqrt{w_i} & \text{if } i \in S, \\[2ex] 2\dfrac{1}{\sqrt{w_i}}\dfrac{\sum_{j\in S} c_{ij}}{-\lambda_{\min}(W^{-1/2}CW^{-1/2})} & \text{if } i \notin S. \end{cases} \tag{16}$$

On the other hand, let $s = (s_1, \ldots, s_n)$ be given by

$$s_i = \begin{cases} 0 & \text{if } i \in S, \\[2ex] 2\dfrac{1}{\sqrt{w_i}}\dfrac{\sum_{j\in S} c_{ij}}{-\lambda_{\min}(W^{-1/2}CW^{-1/2})} - 2\sqrt{w_i} & \text{if } i \notin S. \end{cases}$$

Note that $s \ge 0$ since $-\lambda_{\min}(W^{-1/2}CW^{-1/2}) = -\lambda_{\min}(W^{1/2}QW^{1/2})$ and we are assuming the truthfulness of condition (15). Then, from the definitions of x and s

and taking into account equality (16), we can deduce the following conditions

$$x, s \geq 0, \quad x^T s = 0 \quad \text{and}$$
$$2\left(\frac{W^{-1/2}CW^{-1/2}}{-\lambda_{\min}(W^{-1/2}CW^{-1/2})} + I\right)x = 2\sqrt{w} + s,$$

which are the Karush–Kuhn–Tucker conditions for the problem $P(G, w, C)$. Consequently,

$$\begin{aligned} v(G,w,C) &= 2\sqrt{w}^T x - x^T\left(\frac{W^{-1/2}CW^{-1/2}}{-\lambda_{\min}(W^{-1/2}CW^{-1/2})} + I\right)x \\ &= 2\sqrt{w}^T x - x^T x - \frac{y^T C y}{-\lambda_{\min}(W^{-1/2}CW^{-1/2})} \\ &= 2\alpha(G,w) - \alpha(G,w) - \frac{\sum_{i\in S}(\sum_{j\in S} c_{ij})}{-\lambda_{\min}(W^{-1/2}CW^{-1/2})} \\ &= \alpha(G,w) - 0 = \alpha(G,w). \end{aligned}$$

Conversely, suppose that $\alpha(G, w) = v(G, w, C)$. We first prove that condition (14) is satisfied and that $x = W^{1/2}y$, where y is a characteristic vector of any maximum weight stable set S of (G, w), is an optimal solution of problem $P(G, w, C)$. In fact, computing the objective function value of problem $P(G, w, C)$ for $x = W^{1/2}y$, we obtain

$$\begin{aligned} &2\sqrt{w}^T x - x^T\left(\frac{W^{-1/2}CW^{-1/2}}{-\lambda_{\min}(W^{-1/2}CW^{-1/2})} + I\right)x \\ &\quad = 2\sqrt{w}^T x - x^T x - \frac{x^T W^{-1/2}CW^{-1/2}x}{-\lambda_{\min}(W^{-1/2}CW^{-1/2})} \\ &\quad = \alpha(G,w) - \frac{\sum_{i\in S}\left(\sum_{j\in S} c_{ij}\right)}{-\lambda_{\min}(W^{-1/2}CW^{-1/2})} \leq v(G,w,C) = \alpha(G,w). \end{aligned}$$

Therefore, $\sum_{i\in S}(\sum_{j\in S} c_{ij}) \geq 0$ as $-\lambda_{\min}(W^{-1/2}CW^{-1/2}) > 0$ and hence $\sum_{j\in S} c_{ij} = 0$ for all $i \in S$ since $c_{ii} = 0$ and $c_{ij} \leq 0$ for $ij \notin E$. Thus, condition (14) is verified and x yields an optimal solution of problem $P(G, w, C)$.

In addition, as $x = W^{1/2}y$ solves problem $P(G, w, C)$, the Karush–Kuhn–Tucker conditions imply that

$$2\left(\frac{W^{-1/2}CW^{-1/2}}{-\lambda_{\min}(W^{-1/2}CW^{-1/2})} + I\right)x = 2\sqrt{w} + s,$$

with $s \geq 0$. It follows that, for each $i \notin S$, the corresponding line of the last equality is written as

$$2\frac{1}{\sqrt{w_i}}\frac{\sum_{j\in S} c_{ij}}{-\lambda_{\min}(W^{-1/2}CW^{-1/2})} = 2\sqrt{w_i} + s_i,$$

from which we conclude

$$-\lambda_{\min}(W^{-1/2}CW^{-1/2}) \leq \frac{1}{w_i}\sum_{j\in S} c_{ij},$$

taking into account that $s_i \geq 0$. As, once more, $-\lambda_{\min}(W^{-1/2}CW^{-1/2}) = -\lambda_{\min}(W^{1/2}QW^{1/2})$, condition (15) is valid and the theorem follows. □

Also, as a consequence of Theorem 2, the next result states another sufficient condition for having $\alpha(G,w) = \vartheta'(G,w)$. Although the verification of this sufficient condition needs the computation of $\vartheta(G,w)$, in case it is satisfied the optimal solution of any element of a set of strictly convex quadratic programming problems is shown to yield a maximum weight stable set.

Theorem 4. *Let (G,w) be a weighted graph with at least one edge and Q be an extended weighted adjacency matrix of (G,w) attaining the optimum in* (8). *If $\vartheta'(G,w) > -\lambda_{\min}(W^{1/2}QW^{1/2})$ then $\alpha(G,w) = \vartheta'(G,w)$ and a maximum weight stable set of (G,w) can be obtained by solving, for any $\tau \in]-\lambda_{\min}(W^{1/2}QW^{1/2}), \vartheta'_w]$, the following convex quadratic programming problem*:

$$P(G,w,C,\tau) \;\; \upsilon(G,w,C,\tau) = \max\left\{2\sqrt{w}^T x - x^T\left(\frac{W^{-1/2}CW^{-1/2}}{\tau} + I\right)x : x \geq 0\right\},$$

where $C = WQW$.

Proof. From Case 2 of proof of Theorem 2, we know that if $\vartheta'_w > -\lambda_{\min}(W^{1/2}QW^{1/2})$ the vector $x = W^{1/2}y$ given in (13), with y being the characteristic vector of stable set that supports x, verifies $\sqrt{w}^T x = x^T x = \vartheta'_w$. Hence $\alpha(G,w) \geq w^T y = \sqrt{w}^T W^{1/2} y = \sqrt{w}^T x = \vartheta'_w$, i.e., $\alpha(G,w) = \vartheta'(G,w)$ as required in the theorem's first part.

To see the remaining part, recall that, as it is also stated in Case 2 of proof of Theorem 2, from the orthogonality of columns spaces of $B = (\vartheta'_w)^{-1}xx^T$ and $\vartheta'_w I - \sqrt{w}\sqrt{w}^T + W^{1/2}QW^{1/2}$, it can be concluded that $W^{1/2}QW^{1/2}x = \vartheta'_w(\sqrt{w} - x)$, where x is given in (13). Therefore, if $i \in S$, $x_i = \sqrt{w_i}$ and then

$$\sum_{j\in V}\sqrt{w_i w_j}q_{ij}x_j = 0 \Rightarrow \sum_{j\in V}\frac{\sqrt{w_i w_j}q_{ij}}{\tau}x_j = 0, \quad \forall\, \tau \in]-\lambda_{\min}(W^{1/2}QW^{1/2}), \vartheta'_w],$$

where q_{ij} is the (i,j) entry of matrix Q. Thus, for $i \in S$, the equalities

$$\sum_{j\in V}\frac{\sqrt{w_i w_j}q_{ij}}{\tau}x_j + x_i = \sqrt{w_i} + s_i, \tag{17}$$

where $s_i = 0$, are true for all $\tau \in]-\lambda_{\min}(W^{1/2}QW^{1/2}), \vartheta'_w]$.

On the other hand, if $i \notin S$, $x_i = 0$ and then $W^{1/2}QW^{1/2}x = \vartheta'_w(\sqrt{w} - x)$ implies that

$$\sum_{j \in V} \sqrt{w_i w_j} q_{ij} x_j = \vartheta'_w \sqrt{w_i} \geq \tau \sqrt{w_i}, \quad \forall \tau \in]-\lambda_{\min}(W^{1/2}QW^{1/2}), \vartheta'_w].$$

Hence, for $i \notin S$ there exists $s_i \geq 0$ such that

$$\sum_{j \in V} \frac{\sqrt{w_i w_j} q_{ij}}{\tau} x_j = \sqrt{w_i} + s_i, \quad \forall \tau \in]-\lambda_{\min}(W^{1/2}QW^{1/2}), \vartheta'_w]. \tag{18}$$

From the equalities (17) and (18) we conclude that, for all $\tau \in]-\lambda_{\min}(W^{1/2}QW^{1/2}), \vartheta'_w]$, there exists a vector $s \geq 0$ such that x and s satisfy $x^T s = 0$ as well as the equality

$$\left(\frac{W^{1/2}QW^{1/2}}{\tau} + I \right) x = \sqrt{w} + s.$$

Consequently, for all $\tau \in]-\lambda_{\min}(W^{1/2}QW^{1/2}), \vartheta'_w]$, x satisfies the Karush–Kuhn–Tucker conditions associated with problem $P(G, w, WQW, \tau)$, hence it is an optimal solution of this problem. Since, for those values of τ, $P(G, w, WQW, \tau)$ is a strictly convex quadratic programming problem, solving it allows to obtain its unique solution. This is precisely the vector x given in (13) whose support is a maximum weight stable set of (G, w). $\square$

The following property of weighted graphs verifying the sufficient condition of this last theorem can be asserted.

Corollary 4.1. *Let (G, w) be a weighted graph verifying the conditions of Theorem 4. If S is a maximum weight stable set of (G, w) then, for all $i \notin S$,*

$$\frac{1}{w_i} \sum_{j \in S} c_{ij} = \vartheta'(G, w), \quad \forall i \notin S, \tag{19}$$

where c_{ij} denotes the entry (i, j) of $C = WQW$. Consequently, the right-hand side of (15) equals $\vartheta'(G, w)$.

Proof. Consider once more the equality $W^{1/2}QW^{1/2}x = \vartheta'_w(\sqrt{w} - x)$ deduced in proof of Theorem 2, where $x = W^{1/2}y$ is given in (13), with y being the characteristic vector of stable set S that supports x. Hence $(W^{1/2}QW^{1/2} + \vartheta'_w I)W^{1/2}y = \vartheta'_w \sqrt{w}$ and, multiplying this equality on the left by $W^{1/2}$, we obtain $(C + \vartheta'_w W)y = \vartheta'_w w$. So, for $i \notin S$, row i of this system can be written as

$$\sum_{j \in S} c_{ij} = \vartheta'_w w_i,$$

since row i of Wy is null. Consequently, the corollary follows. $\square$

The above corollary allows to conclude that the weighted graphs verifying the sufficient condition of Theorem 4 also satisfy (15). This is mandatory since these

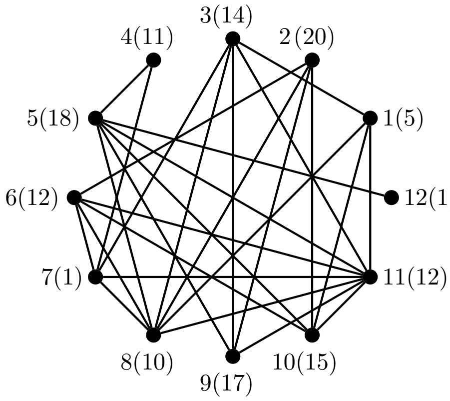

Fig. 1. A weighted graph (G, w) (the weights in brackets) for which $\alpha(G, w) = \vartheta'(G, w)$.

graphs satisfy $\alpha(G, w) = \vartheta'(G, w)$. As an example, consider the weighted graph (G, w) depicted in Fig. 1 where $w = (5, 20, 14, 11, 18, 12, 1, 10, 17, 15, 12, 1)^T$. We have $\vartheta'(G, w) = 54 > 53.80 = -\lambda_{\min}(W^{1/2}QW^{1/2})$, with Q being an extended weighted adjacency matrix of G such that $\vartheta'(G, w) = \lambda_{\max}(\sqrt{w}\sqrt{w}^T - W^{1/2}QW^{1/2})$. Consequently, by Theorem 4, $\alpha(G, w) = \vartheta'(G, w)$, the vector

$$x = (0, 0, 0, 3.32, 0, 0, 0, 3.16, 4.12, 3.87, 0, 1)^T$$

solves problem $P(G, w, C)$ (where $C = WQW$), verifies $\sqrt{w}^T x = 54 = \vartheta'(G, w)$ and its support $S = \{4, 8, 9, 10, 12\}$ is a maximum weight stable set S of (G, w).

On the other hand, by Corollary 4.1, we have that $\frac{1}{w_i}\sum_{j\in S} c_{ij} = \vartheta'(G, w) = 54$, for all $i \notin S$. Therefore, (15) holds since $-\lambda_{\min}(W^{1/2}QW^{1/2}) = 53.80$. As the equalities $\sum_{j\in S} c_{ij} = 0$, for all $i \in S$, are satisfied for this example, Theorem 3 also implies that $\alpha(G, w) = \vartheta'(G, w)$.

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